

# Master Integrals, Superintegrability and Quadratic Algebras

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## Abstract

In this paper we use a generalization of Oevel’s theorem about master symmetries to relate them with superintegrability and quadratic algebras.

## I Introduction

In this article a general framework is built in terms of master symmetries and recursion operator to provide superintegrability and quadratic algebras.

This is applied to the isotropic harmonic oscillator, the rational Calogero-Moser system and the “Goldfish” model.

## II Background

Let  $M$  be a differentiable ( $C^\infty$ ) manifold of finite dimension and  $\Lambda$  a bivector (a 2-times contravariant skew-symmetric tensor field) on  $M$ . Associated with  $\Lambda$  there is a natural

morphism  $\Lambda^\sharp$  from the cotangent bundle  $T^*M$  into the tangent bundle  $TM$  defined, for all  $\alpha, \beta \in T^*M$ , by

$$\langle \Lambda^\sharp(\alpha), \beta \rangle = \Lambda(\alpha, \beta), \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual coupling between 1-forms and vector fields.

We also define a bilinear map from  $C^\infty(M) \times C^\infty(M)$  into  $C^\infty(M)$  by

$$\{f, g\} = \Lambda(df, dg), \quad f, g \in C^\infty(M). \quad (2.2)$$

Due to the properties of  $\Lambda$ , this bracket satisfies

**PB1**  $\{f, g\} = -\{g, f\}$  *skew-symmetry*

**PB2**  $\{fh, g\} = f\{h, g\} + \{f, h\}g$  *Leibniz rule*

We say that  $(M, \Lambda)$  is a *Poisson manifold*, and  $\Lambda$  is a *Poisson tensor*, if, in addition, the bracket (2.2) satisfies the *Jacobi identity*

**PB3**  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$

which is equivalent to the vanishing of the Schouten-Nijenhuis bracket  $[\Lambda, \Lambda]$ .

We call a vector field  $X$  an *infinitesimal Poisson automorphism* if it satisfies

$$X\{f, g\} = \{X(f), g\} + \{f, X(g)\}, \quad f, g \in C^\infty(M).$$

Given a differentiable function  $H$  on  $M$  the *Hamiltonian vector field associated with  $H$*  is the vector field defined by

$$X_H(x) = \Lambda^\sharp(x)(dH(x)), \quad x \in M. \quad (2.3)$$

If  $\Lambda$  is a Poisson tensor then

$$[X_f, X_g] = [\Lambda^\sharp(df), \Lambda^\sharp(dg)] = \Lambda^\sharp(d\{f, g\}) = X_{\{f, g\}} \quad (2.4)$$

that is,  $\Lambda^\sharp$  is a Lie algebra homomorphism between the Lie algebra of differentiable functions  $(C^\infty(M), \{.,.\})$  and the Lie algebra of vector fields  $(A(M), [.,.])$ .

An integral of motion of  $H$ , or of  $X_H$ , is a differentiable function  $F$  such that

$$\{H, F\} = X_H(F) = 0, \quad (2.5)$$

so is constant along the orbits of the Hamiltonian system  $X_H$

$$\frac{d\phi}{dt} = X_H(\phi). \quad (2.6)$$

By a *Nijenhuis operator*  $R$  in a manifold  $M$  we mean a  $(1,1)$ -tensor satifying, for all vector fields  $Z$  in  $M$ ,

$$\mathcal{L}_{R(Z)}R = R\mathcal{L}_ZR. \quad (2.7)$$

The Nijenhuis operators transform closed 1-forms into closed 1-forms in the following sense

**Proposition 1** *Let  $R$  be a Nijenhuis operator and  $\alpha$  a closed 1-form such that  $\alpha_1 = {}^tR\alpha$  is also closed then, for all  $i \in \mathbb{N}$ ,  $\alpha_i = {}^tR^i\alpha_1$  (where  ${}^tR^i$  means  $i^{th}$  iterates of  ${}^tR$ ) are closed.*

By a conformal vector field of a tensor  $W$  we mean a vector field  $Z$  such that  $\mathcal{L}_ZW = cW$ , for some constant  $c \in \mathbb{R}$ .

With Oevel's Theorem [11], a Nijenhuis operators helps to define new symmetries if a conformal vector field is known.

**Theorem 2 (Oevel)** *Let  $R$  be a recursion operator of  $X_0 \in A(M)$ , this means  $\mathcal{L}_X R = 0$ , and  $Z_0 \in A(M)$  a conformal vector field of  $X_0$  and  $R$  such that*

$$\mathcal{L}_{Z_0}X_0 = \lambda X_0, \quad \mathcal{L}_{Z_0}R = \mu R, \quad \lambda, \mu \in \mathbb{R}.$$

*If  $R$  is also a Nijenhuis operator then, defining the sequences  $X_n = R^n X_0$  and  $Z_n = R^n Z_0$ ,  $n \in \mathbb{N}$ , we have, for all  $n, m \in \mathbb{N}_0$*

$$\mathcal{L}_{Z_n}R = \mu R^{n+1},$$

$$[Z_n, Z_m] = \mu(m - n)Z_{n+m}$$

and

$$[Z_n, X_m] = (\lambda + m\mu)X_{n+m}.$$

The  $Z_i$ 's are called *master symmetries* or *symmetries of second order* of the vector fields  $X_j$  because  $[[Z_i, X_j], X_j] = 0$  but  $[Z_i, X_j] \neq 0$  and they help us define new symmetries of the system. We call *master integrals* to functions  $G$  which may not be integrals of motion of the system but they induce an integral  $X(G)$ .

Now let us consider two linearly independent Poisson structures  $\Lambda_0$  and  $\Lambda_1$  in  $M$ . We say that they are compatible if  $\Lambda_0 + \Lambda_1$  is again a Poisson tensor. The compatibility condition is equivalent to the vanishing of the Schouten-Nijenhuis bracket  $[\Lambda_0, \Lambda_1]$ .

A vector field  $X \in A(M)$  is said to be *bihamiltonian* if it is Hamiltonian with respect to two independent compatible Poisson tensors, that is, if there exist two functions  $H, F \in C^\infty(M)$  such that

$$X = \Lambda_0^\sharp(dH) = \Lambda_1^\sharp(dF). \quad (2.8)$$

There is an important example of bihamiltonian manifold: the *Poisson-Nijenhuis manifold* [9]. This manifold is special because one of the Poisson tensors is obtain from the other by means of a *Nijenhuis operator*. For instance, a bihamiltonian manifold  $(M, \Lambda_0, \Lambda_1)$  such that the Poisson tensor  $\Lambda_0$  is non-singular (symplectic manifold), is a Poisson-Nijenhuis manifold with Nijenhuis operator  $R = \Lambda_1^\sharp \Lambda_0^{\sharp-1}$ . If  $X_1 = \Lambda_0^\sharp(dH_1) = \Lambda_1^\sharp(dH_0)$  is a bihamiltonian system then we may define a sequence of symmetries of  $X$ ,  $X_i = R^{i-1}X_1$  and, if the first cohomology group is trivial, a sequence of integrals of motion in involution  $(H_i)_{i \in \mathbb{N}}$  such that  $X_i = \Lambda_0^\sharp(dH_i)$ , ie  $dH_i = {}^tR^i(dH_0)$ ,  $i \in \mathbb{N}$ .

### III The superintegrability and the cubic algebra

Let  $X$  be a vector field on a manifold  $M$  of dimension  $n$ . It is called *maximally superintegrable* if it possesses  $n - 1$  functionally independent first integrals.

There are several examples of maximally superintegrable systems, some of them are shown in the last section.

The superintegrability may be a consequence of the existence of a sufficient number of master integrals, as the next proposition shows.

**Proposition 3** *Let  $X$  be a vector field on a manifold  $M$  and  $G, F \in C^\infty(M)$  master integrals of  $X$ . Then the function*

$$L = X(G)F - X(F)G \quad (3.9)$$

*is an integral of the vector field  $X$ .*

**Proof:**

Just notice that

$$\begin{aligned} X(L) &= X(X(G))F + X(G)X(F) - X(X(F))G - X(F)X(G) \\ &= X(X(G))F - X(X(F))G = 0, \end{aligned}$$

because that  $X(F)$  and  $X(G)$  are integrals of the system. ■

**Remark 4** *If  $X$  is a vector field on a manifold of dimension  $2n$  and if  $n$  functionally independent master integrals are known  $G_1, \dots, G_n$  then we can define the integrals of motion  $F_i = X(G_i)$  and  $L_{i,j} = X(G_i)G_j - G_iX(G_j)$  which may provide the superintegrability of the system if  $2n - 1$  of them are functionally independent.*

**Theorem 5** *Let  $X$  be a Poisson infinitesimal automorphism on a Poisson manifold  $(M, \{.,.\})$ . Suppose there exist master integrals of  $X$ ,  $G_i$ , such that  $\{G_i, X(G_i), i \in J \subset \mathbb{N}\}$  is a basis of a Lie subalgebra of  $(C^\infty(M), \{.,.\})$*

*Then, for all  $i, j \in \mathbb{N}$  the functions  $X(G_i)$  and  $L_{G_i, G_j} = X(G_i)G_j - X(G_j)G_i$  generate a cubic algebra for the Poisson bracket.*

**Proof:**

Once  $\{G_i, X(G_i), i \in J\}$  generates a Lie algebra, for each  $i, j, k \in J$  there exist constants  $a_{i,j}^k, b_{i,j}^k$  such that

$$\{G_i, G_j\} = \sum_{k \in J} (a_{i,j}^k G_k + b_{i,j}^k X(G_k)). \quad (3.10)$$

Applying  $X$  twice to the last equation we obtain

$$X(\{X(G_i), (G_j)\} + \{G_i, X(G_j)\}) = X(\sum_{k \in J} a_{i,j}^k X(G_k)),$$

so

$$\{X(G_i), X(G_j)\} = 0.$$

Writing  $\{G_i, X(G_j)\} = \sum_{k \in J} (c_{i,j}^k X(G_k) + d_{i,j}^k G_k)$ , with  $c_{i,j}^k, d_{i,j}^k$  constants, and noticing that

$$X(\{G_i, X(G_j)\}) = \{X(G_i), X(G_j)\} = 0,$$

we have  $\sum_{k \in J} d_{i,j}^k X(G_k) = 0$ , which yields  $d_{i,j}^k = 0$ .

Thus

$$\begin{aligned} \{L_{G_i, G_j}, X(G_k)\} &= X(G_i)\{G_j, X(G_k)\} - X(G_j)\{G_i, X(G_k)\} \\ &= \sum_l [c_{j,k}^l X(G_i)X(G_l) - c_{i,k}^l X(G_j)X(G_l)] \end{aligned}$$

and

$$\begin{aligned}
\{L_{G_i, G_j}, L_{G_k, G_h}\} &= \\
&= X(G_k)[L_{\{G_i, G_h\}, G_j} - L_{\{G_j, G_h\}, G_i}] + X(G_h)[L_{\{G_j, G_k\}, G_i} - L_{\{G_i, G_k\}, G_j}] \\
&\quad + L_{G_j, G_k}\{G_i, X(G_h)\} + L_{G_i, G_h}\{G_j, X(G_k)\} \\
&\quad - L_{G_j, G_h}\{G_i, X(G_k)\} - L_{G_i, G_k}\{G_j, X(G_h)\} \\
&= \sum_l [X(G_k)(a_{i,h}^l L_{G_l, G_j} + b_{i,h}^l L_{X(G_l), G_j} - a_{j,h}^l L_{G_l, G_i} - b_{j,h}^l L_{X(G_l), G_i}) \\
&\quad + X(G_h)(a_{j,k}^l L_{G_l, G_i} + b_{j,k}^l L_{X(G_l), G_i} - a_{i,k}^l L_{G_l, G_j} - b_{i,k}^l L_{X(G_l), G_j}) \\
&\quad + c_{i,h}^l L_{G_j, G_k} X(G_l) + c_{j,k}^l L_{G_i, G_h} X(G_l) - c_{i,k}^l L_{G_j, G_h} X(G_l) - c_{j,h}^l L_{G_i, G_k} X(G_l)] \\
&= \sum_l [X(G_k)(a_{i,h}^l L_{G_l, G_j} - a_{j,h}^l L_{G_l, G_i}) + X(G_h)(a_{j,k}^l L_{G_l, G_i} - a_{i,k}^l L_{G_l, G_j}) \\
&\quad + c_{i,h}^l L_{G_j, G_k} X(G_l) + c_{j,k}^l L_{G_i, G_h} X(G_l) - c_{i,k}^l L_{G_j, G_h} X(G_l) - c_{j,h}^l L_{G_i, G_k} X(G_l) \\
&\quad + (b_{i,h}^l X(G_j) - b_{j,h}^l X(G_i))X(G_l)X(G_k) + (b_{j,k}^l X(G_i) - b_{i,k}^l X(G_j))X(G_l)X(G_h)]
\end{aligned}$$

This last expression being cubic in the quantities  $X(G_i)$  and  $L_{G_i, G_j}$ , we refer to this result as that they generate a cubic algebra. ■

**Corollary 6** *If for each  $i, j \in \mathbb{N}$ ,  $\sum_l b_{i,j}^l X(G_l) = b_i X(G_j) - b_j X(G_i)$  or all the constants  $b_{i,j}^k$  are zero then the integrals of motion generate a quadratic algebra.*

**Remark 7** *Notice that in the above proposition we could have just demanded that  $d\{G_i, G_j\}$  be a linear combination of the  $dG$ 's and the  $dX(G)$ 's.*

**Theorem 8 (Generalization of Oevel's theorem)** *Let  $X$  be a vector field on a manifold  $M$ ,  $R$  a Nijenhuis operator which is also a recursion operator of  $X$  and  $P$  a  $(1,1)$ -tensor satisfying*

$$\mathcal{L}_X P = a(R) \tag{3.11}$$

and

$$\mathcal{L}_{PX} R = b(R), \tag{3.12}$$

with  $a(R)$ ,  $b(R)$  polynomials in  $R$ . Then, defining the sequences  $X_i = R^i X$ ,  $Y_i = R^i(PX)$ ,  $i \in \mathbb{N}_0$ , we have

$$[X_i, X_j] = 0, \tag{3.13}$$

$$[X_i, Y_j] = a(R)(X_{i+j}) - ib(R)(X_{i+j-1}) \quad (3.14)$$

$$[Y_i, Y_j] = (j - i)b(R)Y_{i+j-1}. \quad (3.15)$$

**Proof:**

The proof is similar to the original Oevel's theorem proof, which can be seen at [11]. ■

Suppose that  $M$  has trivial first cohomology group and is endowed with a non-degenerated Poisson structure  $\Lambda$  such that  $R\Lambda^\sharp = \Lambda^{\sharp t}R$  (this means that the tensor  $R\Lambda$  is a bivector). Furthermore suppose that there exist functions such that  $X = \Lambda^\sharp(dH_1) = R\Lambda^\sharp(dH_0)$  and  $Y = \Lambda^\sharp(dG_1) = R\Lambda^\sharp(dG_0)$ .

Then Proposition 1 ensures us that the 1-forms

$$\alpha_i = {}^tR^i(dH_1), \quad \beta_i = {}^tR^i(dG_1), \quad i \in \mathbb{N}$$

are closed and we can consider them exact because of the triviality of the first cohomology group.

Write  $\alpha_i = dH_i$  and  $\beta_i = dG_i$ , for all  $i \in \mathbb{N}$ .

First notice that  $R\Lambda$  being a bivector yields

$$\begin{aligned} X_i(H_j) &= \langle X_i, dH_j \rangle = \langle X_{i+j}, dH_1 \rangle \\ &= R^{i+j}\Lambda^\sharp(dH_1, dH_1) = 0. \end{aligned}$$

Moreover (3.14) ensures that the  $G$ 's are master integrals of the  $X$ 's because

$$\begin{aligned} X_i(X_i(G_j)) &= X_i(\{H_i, G_j\}) = -[X_i, Y_j](H_i) \\ &= (ib(R)X_{i+j-1} - a(R)X_{i+j})(H_i) = 0, \end{aligned}$$

relation (3.15) implies

$$d\{G_i, G_j\} = (j - i)b({}^tR)dG_{i+j-1}$$

and

$$\begin{aligned} \{X_i(G_j), G_k\} &= -Y_k(X_i(G_j)) = [X_i, Y_k](G_j) - X_i(Y_k(G_j)) \\ &= ib(R)X_{i+k-1}(G_j) - a(R)X_{i+k}(G_j) - X_i(\{G_k, G_j\}) \\ &= ib(R)X_i(G_{j+k-1}) - a(R)X_i(G_{j+k}) - (j - k)b(R)X_i(G_{k+j-1}) \\ &= (i + k - j)b(R)X_i(G_{j+k-1}) - a(R)X_i(G_{j+k}). \end{aligned}$$

So  $\{X_i(G_j), G_k\}$  can be written as a linear combination of the  $X_i(G)$ 's.

Now we can apply Theorem 5, with the  $b$ 's equal to zero, and guarantee that, for each  $i \in \mathbb{N}$  the integrals of  $X_i$ ,  $X_i(G_j)$  and  $L_{k,j}^i = X_i(G_k)G_j - X_i(G_j)G_k$ ,  $j, k \in \mathbb{N}_0$ , close quadratically under the Poisson bracket.

Futhermore notice that

$$[R\Lambda, \Lambda](\Lambda^{\sharp-1}Y) = \mathcal{L}_{RY}\Lambda + (\mathcal{L}_Y\Lambda) \circ {}^tR + (\mathcal{L}_Y R) \circ \Lambda$$

so, as  $Y$  and  $RY$  are Hamiltonian vector fields, we have

$$[R\Lambda, \Lambda](\Lambda^{\sharp-1}Y) = (\mathcal{L}_Y R) \circ \Lambda = b(R)\Lambda.$$

Thus, if  $R\Lambda$  is a Poisson tensor then it is compatible with  $\Lambda$ , because  $R$  is a Nijenhuis operator, and  $b(R)\Lambda = 0$ .

But this implies that

$$b(R)X_i = b(R)\Lambda^{\sharp}(dH_i) = 0 \text{ and } b(R)Y_i = b(R)\Lambda^{\sharp}(dG_i) = 0,$$

so the relations (3.13), (3.14) and (3.15) become

$$[X_i, X_j] = [Y_i, Y_j] = 0; \quad [X_i, Y_j] = a(R)X_{i+j}$$

and

$$d\{G_i, G_j\} = 0; \quad d\{G_k, X_i(G_j)\} = a(R)X_i(G_{j+k}).$$

Although in the last procedure we need two master integrals, Hamiltonians of the master symmetries, to generate a new sequence of integrals of motion, we may construct it only knowing one master integral of all the vector fields.

**Proposition 9** *Under the conditions of proposition 8, suppose there exists a master integral  $G$  of all the vector fields  $X_i$ ,  $i \in \mathbb{N}_0$ , then the functions  $G_i = Y_i(G)$ ,  $i \in \mathbb{N}_0$  are also master integrals of the same vector fields and, for each  $k \in \mathbb{N}_0$ ,*

$$L_{i,j}^k = X_k(G_i)G_j - X_k(G_j)G_i, \text{ for all } i, j \in \mathbb{N}_0,$$

*are integrals of  $X_k$ .*



**Proof:**

Due to relation (3.14), we have, for all  $i, k \in \mathbb{N}_0$

$$\begin{aligned} X_k(G_i) &= [X_k, Y_i](G) + Y_i(X_k(G)) \\ &= a(R)X_{i+k}(G) - kb(R)X_{i+k-1}(G) + Y_i(X_k(G)). \end{aligned}$$

But

$$X_k(Y_i(X_k(G))) = a(R)X_{i+k}(X_k(G)) - kb(R)X_{i+k-1}(X_k(G)) + Y_i(X_k(X_k(G))) = 0,$$

because  $X_k(G)$  is an integral of all the  $X_j$ , so  $Y_i(X_k(G))$  is an integral of  $X_k$  and  $X_k(G_i)$  is then an integral of  $X_k$ . Thus  $G_i$  is a master integral of all the  $X_k$ . ■

**Remark 10** *It seems that in the previous procedure only one Hamiltonian  $G_0$  would be necessary, but note that the procedure applied to  $G_0$  yields  $G_i = Y_i(G_0) = R^i \Lambda(dG_0, dG_0) = 0$ , i.e. all master integrals are zero.*

## IV Examples

### Example 11 (Isotropic Harmonic Oscillator and the Fernandes' Theorem)

The Isotropic Harmonic Oscillator is the Hamiltonian system in  $(\mathbb{R}^{2n}, (q_i, p_i))$ , defined by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^n \frac{1}{2}(p_i^2 + q_i^2). \quad (4.16)$$

It is well known that this system is completely integrable with constants of motion in involution

$$E_i = \frac{1}{2}(p_i^2 + q_i^2), \quad i = 1, \dots, n, \quad (4.17)$$

and it is bi-Hamiltonian [10] with respect to the compatible Poisson tensors

$$\Lambda_0 = \sum_{i=1}^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \quad (4.18)$$

and

$$\Lambda_1 = \sum_{i=1}^n E_i \left( \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \right). \quad (4.19)$$

The Hamiltonian vector field can be expressed as

$$X_{\mathcal{H}} = \sum_{i=1}^n (p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}) = \Lambda_0^{\sharp}(d\mathcal{H}) = \Lambda_1^{\sharp}(dH_0), \quad (4.20)$$

with  $H_0 = \ln(E_1) + \dots + \ln(E_n)$ . Defining the recursion operator of the system

$$R = \Lambda_1^{\sharp} \Lambda_0^{\sharp-1} = \sum_{i=1}^n E_i (\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i), \quad (4.21)$$

a sequence of Hamiltonians and of Hamiltonian vector fields can be defined

$$X_i = N^{i-1} X_{\mathcal{H}} = \Lambda_0^{\sharp}(dH_i), \quad i = 1, \dots, n. \quad (4.22)$$

Notice that the  $(1, 1)$ -tensor

$$P = \sum_{i=1}^n \varphi_i (\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i), \quad (4.23)$$

with  $\varphi_i = \arcsin(\frac{q_i}{\sqrt{q_i^2 + p_i^2}})$ , satisfies the conditions of Theorem 8 with  $a(R) = Id$  and  $b(R) = 0$ . So defining the sequence

$$Y_k = R^k P X = \sum_{i=1}^n \varphi_i E_i^k (p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}), \quad (4.24)$$

we have

$$\begin{aligned} [Y_i, Y_j] &= 0 \\ [Y_i, X_j] &= -X_{i+j} \end{aligned}$$

and also

$$\begin{aligned} dZ_i(H_j) &= 0 \\ [Y_i, R^j \Lambda_0] &= -R^{i+j} \Lambda_0 \end{aligned}$$

i.e., the vector fields  $Y_i$  are not Hamiltonian with respect to any of the Poisson structures.

But let us define the functions  $G = \sum_{i=1}^n E_i \varphi_i$  and notice that

$$X_i(G) = \sum_k E_k^{i-1} (p_k \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial p_k})(G) = iH_i. \quad (4.25)$$

So the functions  $G_i = Y_i(G) = \sum_k \varphi_k E_k^{i+1}$  satisfy

$$X_i(G_j) = X_{i+j}(G) = (i+j)H_{i+j},$$

$$\begin{aligned}
\{X_i(G_j), G_k\} &= \{(i+j)H_{i+j}, G_k\} \\
&= (i+j)X_{i+j}(G_k) = (i+j)X_{i+j+k}(G) \\
&= (i+j)(i+j+k)H_{i+j+k}
\end{aligned}$$

and

$$\begin{aligned}
\{G_i, G_j\} &= \Lambda_0(dG_i, dG_j) = \Lambda_0(dZ_i(G), dZ_j(G)) \\
&= \Lambda_0(d(\sum_{k=1}^n E_k^{i+1}\varphi_k), d(\sum_{h=1}^n E_h^{j+1}\varphi_h)) = (i-j)Z_{i+j}(G) = (j-i)G_{i+j}.
\end{aligned}$$

Therefore, for each  $k \in \mathbb{N}_0$  the integrals of motions of  $X_k$ ,  $X_k(G_j)$ 's, and  $L_{i,j}^k = X_k(G_i)G_j - X_k(G_j)G_i = (i+k)H_{i+k}G_j - (j+k)H_{j+k}G_i$  close quadratically under the Poisson bracket defined by  $\Lambda_0$ .

Now let us consider a little more general configuration, in which the isotropic harmonic oscillator is a particular case.

Given a completely integrable Hamiltonian system  $(M^{2n}, \omega, H)$  in a symplectic manifold, Fernandes [5] establishes necessary and sufficient conditions for the existence of a second Poisson structure giving the complete integrability of the system, in a neighborhood of a fixed invariant torus. Without loss of generality let us consider

$$(M^{2n} = \mathbb{R}^n \times \mathbb{T}^n, (s_i, \theta_i)_{i=1}^n), \quad H = H(s_1, \dots, s_n) \text{ and } \omega = \sum_i ds_i \wedge d\theta_i. \quad (4.26)$$

**Definition 12** *Let  $(x^1, \dots, x^{n+1})$  be affine coordinates in a  $(n+1)$ -dimensional affine space  $A^{n+1}$ . A hypersurface in  $A^{n+1}$  is called a hypersurface of translation if it admits a parametrization of the form*

$$(y^1, \dots, y^n) \rightarrow x^j(y^1, \dots, y^n) = a_1^j(y_1) + \dots + a_n^j(y_n), \quad (j = 1, \dots, n+1). \quad (4.27)$$

**Theorem 13 ([5])** *The completely integrable Hamiltonian system (4.26) admits a second Poisson structure, giving its complete integrability if and only if the graph of the Hamiltonian function is a hypersurface of translation relative to the affine structure determined by the action variables.*

We present the “only if” part of the proof because in what follows the new Poisson structure will be needed.

**Proof:**

Assume the  $(M^{2n}, \omega, H)$  is a completely integrable system and that the graph of  $H$  is a hypersurface of translation relative to the action variables  $(s^i)$ , so it has a parametrization of the form (4.27) with  $x^i = s^i$ ,  $i = 1, \dots, n$  and  $x^{n+1} = H$ . We can choose the parameters  $(y^i)$  so that the Hamiltonian takes the simple form

$$H(y^1, \dots, y^n) = y^1 + \dots + y^n.$$

If  $(\varphi^1, \dots, \varphi^n)$  are the coordinates conjugated to  $(y^1, \dots, y^n)$ , we define a second Poisson structure by the formula

$$\Lambda_1 = \sum_{i=1}^n y^i \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial \varphi^i}.$$

One checks easily that the two Poisson structures are compatible, and that the recursion operator is given by

$$R = \sum_{i=1}^n y^i \left( \frac{\partial}{\partial y^i} \otimes dy^i + \frac{\partial}{\partial \varphi^i} \otimes d\varphi^i \right).$$

It is now clear from the expression of the Hamiltonian function in the  $y$ -coordinates that  $\mathcal{L}_{X_H} R = 0$ , so the vector field  $X_H$  is bi-Hamiltonian. ■

Thus a completely Hamiltonian system whose Hamiltonian's graph is a hypersurface of translation relative to the affine structure determined by the action variables, is bi-Hamiltonian

$$X_H = \Lambda_0^\sharp(dH) = \Lambda_1^\sharp(dH_0)$$

with  $H_0 = \sum_{i=1}^n \ln(y^i)$ .

The  $(1, 1)$ -tensor

$$P = \sum_{i=1}^n \left( \varphi^i \frac{\partial}{\partial \varphi^i} \otimes d\varphi_i + \frac{\partial}{\partial y_i} \otimes dy_i \right)$$

satisfies

$$\mathcal{L}_{X_H} P = Id \text{ and } \mathcal{L}_{P_X} R = 0$$

so we can apply Theorem 8 and conclude that the vector fields  $Y_k = R^k P X = \sum_{i=1}^n y_k \varphi^i \frac{\partial}{\partial \varphi^i}$

and the function  $G = \sum_{i=1}^n y^i \varphi^i$  allows us to define the sequence of functions  $G_i = Y_i(G)$  ( $i =$

$0, 1, \dots$ ), such that

$$\begin{aligned} X_i(G) &= iH_i, \\ X_i(G_j) &= X_{i+j}(G) = (i+j)H_{i+j}, \\ \{X_i(G_j), G_k\} &= (i+j)X_i(G_{k+j}) \end{aligned}$$

and

$$\{G_i, G_j\}_0 = (i-j)G_{i+j}.$$

Thus, for each  $i \in \mathbb{N}$ , the integrals of motion of  $X_i$ ,  $L_{k,j}^i = X_i(G_k)G_j - X_i(G_j)G_k$  and  $X_i(G_j)$ , close quadratically under the Poisson bracket defined by  $\Lambda_0$ .

#### Example 14 (The Rational Calogero-Moser System)

The rational Calogero-Moser system is an integrable Hamiltonian system defined by

$$\mathcal{H} = \sum_{i=1}^n \left( \frac{p_i^2}{2} + \frac{g^2}{2} \sum_{j \neq i} (q_i - q_j)^{-2} \right). \quad (4.28)$$

It admits the pair of matrices  $(L, M)$ ,

$$L_{ij} = p_i \delta_i^j + g \sqrt{-1} (q_i - q_j)^{-1} (1 - \delta_i^j), \quad (4.29)$$

$$M_{ij} = g \sqrt{-1} \sum_{h \neq i} (q_i - q_h)^{-2} \delta_i^j - g \sqrt{-1} (q_i - q_j)^{-2} (1 - \delta_i^j) \quad (4.30)$$

as a Lax pair.

The Hamiltonian vector field with respect to the canonical Poisson structure in  $R^{2n}$  is

$$X_H = \sum_i \left( p_i \frac{\partial}{\partial q_i} + \sum_{j \neq i} 2(q_i - q_j)^{-3} \frac{\partial}{\partial p_i} \right). \quad (4.31)$$

This system is completely integrable when we consider the sequence of integrals of motion  $F_i = \text{Tr}(L^i)$ ,  $i = 1, \dots, n$ .

Moreover, following [12], if we consider the functions  $G_i = \text{Tr}(QL^{i-1})$ , which provide the algebraic linearization of the system [2],[3], the Hamiltonian vector field becomes

$$X_1 = \sum_i F_i \frac{\partial}{\partial G_i} \quad (4.32)$$

and we may define the following compatible Poisson tensors [10]

$$\Lambda_0 = \sum_i \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i}, \quad (4.33)$$

$$\Lambda_1 = \sum_i F_i \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i}. \quad (4.34)$$

The system  $\dot{u} = X_1(u)$  is bi-Hamiltonian with respect to these Poisson structures and the bi-Hamiltonian sequence of integrals of motion is

$$\Lambda_0^\sharp(dh_j) = \Lambda_1^\sharp(dh_{j-1}), \quad j = 0, 1, \dots \quad (4.35)$$

where

$$h_{-1} = \ln(F_1 \dots F_n),$$

$$h_j = \frac{1}{2(j+1)} \text{Tr}(\Lambda_1^\sharp \Lambda_0^{\sharp-1})^{j+1} = \frac{1}{j+1} \sum_i (F_i)^{j+1}, \quad j = 0, 1, \dots \quad (4.36)$$

Notice that  $X_1 = \Lambda_0^\sharp(dh_1) = \Lambda_1^\sharp(dh_0)$  and if we define the sequence of Hamiltonian vector fields

$$X_i = (\Lambda_1^\sharp \Lambda_0^{\sharp-1})^{i-1} X_1 = \sum_k (F_k)^i \frac{\partial}{\partial G_k} \quad i = 1, 2, \dots, \quad (4.37)$$

the following relation holds

$$X_i(\sum_k G_k F_k) = (i+1)h_i. \quad (4.38)$$

Now consider the  $(1, 1)$ -tensor

$$P = \sum_{i=1}^n \frac{\partial}{\partial F_i} \otimes dG_i$$

that satisfies

$$\mathcal{L}_{PX}R = R \text{ and } \mathcal{L}_XP = Id$$

and define the sequence of master symmetries

$$Y_i = (\Lambda_1^\sharp \Lambda_0^{\sharp-1})^i PX_1. \quad (4.39)$$

Considering the functions  $g_i = Z_i(\sum_k (G_k F_k)) = \sum_k F_k^{j+1} G_k$  we have

$$X_i(g_j) = \sum_k F_k^{i+j+1},$$

$$\{X_i(g_j), g_k\}_0 = (i+j+1)X_i(g_{j+k})$$

and

$$\{g_i, g_j\}_0 = (i - j)g_{i+j}.$$

Now the Propositions 5 and 9 ensure that for each  $X_i$ ,  $i \in \mathbb{N}$ , the integrals  $X_i(g_j)$  and  $L_{k,j}^i = X_i(g_k)g_j - X_i(g_j)g_k$  close quadratically under  $\{.,.\}_0$ .

**Example 15 (The Goldfish System)** The Goldfish system was first introduced by Calogero [1] and is the Hamiltonian system in  $(\mathbb{R}^{2n}, (q_i, p_i))$  defined by  $X = \Lambda_0^\sharp(dH)$ , where  $\Lambda_0$  is the canonical Poisson tensor and

$$H = \sum_{i=1}^n \frac{g_i(q_i)}{\prod_{j \neq i} (q_i - q_j)} e^{ap_i}, \quad (4.40)$$

with  $g_i$  arbitrary smooth functions, each one depending only on the corresponding coordinate  $q_i$  and  $a$  an arbitrary constant.

Defining the Nijenhuis tensor

$$R = q_i \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right) \quad (4.41)$$

and the function

$$K = \sum_{i=1}^n \frac{\rho_i g_i(q_i)}{\prod_{j \neq i} (q_i - q_j)} e^{ap_i}, \quad \rho_i = \prod_{j \neq i} q_j \quad (4.42)$$

this system becomes completely integrable and quasi-bihamiltonian [10]. The functions  $c_i$ , coefficients of the minimal polynomial of  $R$  ( $q^n + \sum_i c_i q^{n-i} = \prod_{i=1}^n (q - q_i)$ ), together with the integrals of motion

$$F_k = \sum_{i=1}^n \frac{\partial c_k}{\partial q_i} \frac{g_i(q_i) e^{ap_i}}{\prod_{j \neq i} (q_i - q_j)} \quad (4.43)$$

linearize algebraically the system because  $\dot{c}_k = aF_k$ .

In the coordinates  $(c_i, F_i)$  the system has a simple bi-Hamiltonian structure defined by the compatible Poisson tensors

$$Q_0 = \sum_{i=1}^n \frac{\partial}{\partial c_i} \wedge \frac{\partial}{\partial F_i}, \quad Q_1 = \sum_{i=1}^n F_i \frac{\partial}{\partial c_i} \wedge \frac{\partial}{\partial F_i} \quad (4.44)$$

and the Hamiltonians

$$H_0 = F_1 + \dots + F_n, \quad H_1 = \frac{1}{2}(F_1^2 + \dots + F_n^2). \quad (4.45)$$

Similarly to last example, the  $(1,1)$ -tensor

$$P = \sum_i \frac{\partial}{\partial F_i} \otimes dc_i$$

and the function  $G = \sum_{i=1}^n F_i c_i$  allow us to define the integrals of motion that close quadratically.

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